REVIEW OF "INTUITIONISTIC ANALYSIS AT THE END OF TIME" BY JOAN RAND MOSCHOVAKIS (2017)

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The paper gives a realizability interpretation of a formal system IC in which classical analysis and Brouwer-style intuitionistic analysis coexist, each brand of analysis coming with its own flavor of sequences, and where the link between the two flavors of sequences is made by an "end of time" axiom recently proposed by Kripke. Using this realizability interpretation, the paper also determines the status of principles for expressing Brouwerian counter-example arguments.

More precisely, IC is the first-order intuitionistic logic with three sorts of variables – number variables $x, y, z, \ldots$, variables for definite sequences $a, b, c, \ldots$, and variables for arbitrary choice sequences $\alpha, \beta, \gamma, \ldots$ – and the mathematical axioms for induction, countable choice, bar induction, continuous choice, the double-negation translation of countable choice for negative formulas containing no arbitrary choice sequences, and the "end of time" axiom,

$$\text{(ET)} \qquad \forall\alpha\neg\neg b\forall x(\alpha(x) = b(x)),$$

that makes a link between classical sequences b and arbitrary choice sequences α. Assuming a stronger version of ET with no double negation would have lead to inconsistency with the assumed continuous choice principle.

Then, given a classical model of classical analysis $\mathcal{M}$ interpreting sequences by members of a set $\mathcal{C}$, a realizability relation is defined between realizers $\epsilon \in \mathcal{C}$ and all formulas of IC. The realizability relation interprets prime formulas by their interpretation in the classical model. A closed formula is realizable when it is realized by some general recursive sequence ϵ. Theorem 4.4 then expresses that ever theorem of IC is realizable; the proof is very precise, giving explicit lambda terms for realizing all of the axioms of IC. As a corollary, one gets a consistency proof for IC relative to the classical model $\mathcal{M}$. In Corollary 4.4.2, a consequences of assuming that $\mathcal{C}$ is (not) the Baire space is shown: one gets that either $\forall\alpha\exists b\forall x(\alpha(x) = b(x))$ becomes realizable, or that $\neg\neg\forall\alpha\exists b\forall x(\alpha(x) = b(x))$ is not realizable.

For arithmetical formulas E, the realizability relation restricted to $\epsilon \in \mathcal{C}$ preserves classical truth: E is realizable with $\epsilon \in \mathcal{C}$ if and only if E is true in $\mathcal{M}$. This is Corollary 4.4.5.

Markov's principle is not realizable. As concerns principles formalizing Brouwer's creating subject arguments:

- The weak Kripke schema (WKS) and its weak weak version (WWKS) are in general not consistent with IC, but WKS is consistent with IC for negative formulas without arbitrary choice sequence variables and WWKS is consistent with IC for closed formulas.
- The independence of premise (IP) schema for negated formulas and therefore Vesley's schema (VS) are consistent with IC.