# REVIEW OF "RATHJEN, MICHAEL: INDEFINITENESS IN SEMI-INTUITIONISTIC SET THEORIES: ON A CONJECTURE OF FEFERMAN"

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In semi-intuitionistic set theories, where the law of excluded middle $\phi \vee \neg\phi$ does not hold for all $\phi$, one can wonder whether it does hold for $\phi$ the Continuum Hypothesis (CH). Feferman had conjectured that CH $\vee\ \neg$CH does not hold (is not provable) for the set theory $\mathbf{T} := \mathbf{SCS} +$ "$\mathbb{R}$ is a set", where $\mathbf{SCS} := \mathbf{IKP} + (\Delta_0{-}\mathrm{LEM}) + (\mathrm{MP}) + (\mathrm{BOS}) + (\mathrm{AC}_{\mathrm{Set}})$, where **IKP** is intuitionistic Kripke-Platek set theory, $\Delta_0-$LEM is the law of excluded middle for $\Delta_0$-formulas, MP is Markov's Principle, BOS is the Bounded Omniscience Schema, and $\mathrm{AC}_{\mathrm{Set}}$ is the Axiom of Choice.

Rathjen first proves that $\Delta_0-$LEM and BOS are redundant in **SCS**, by using Diaconescus well-known argument with $\mathrm{AC}_{\mathrm{Set}}$. He then develops a proof of Feferman's conjecture,

$$\mathbf{T} \nvdash \mathrm{CH} \vee \neg\mathrm{CH},$$

by:

- choosing an appropriate version of relativized constructible hierarchies, $L[A]$;
- defining computability over $L[A]$;
- defining realizability over $L[A]$, with a realizability theorem (Theorem 5.2) showing that from any proof in $\mathbf{T}$ one can effectively construct a heriditarily finite set as a realizer;
- finally, a set $C$ is constructed that provides a contradiction with $L[C]$-realizability of CH $\vee\ \neg$CH.

The paper ends by remarks on extensions of $\mathbf{T}$ that still do not prove CH $\vee\ \neg$CH, as well as showing an analogous result of Projective Determinacy (PD): a proof that $\mathbf{T} \nvdash \mathrm{PD} \vee \neg\mathrm{PD}$.

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